ON AN INTEGRAL CRITERION FOR STABILITY OF MOTION

(OB ODNOM INTEGRAL'NOM PRIZNAKE USTOICHIVOSTI DVIZHENIIA)

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It is pointed out that in the solution of problems on the self-synchronization of identical mechanical vibrators the result of the investigation of the stability of the synchronized motion is obtained in a form which is similar to a certain integral criterion of stability. It is shown by a number of examples that the conditions of stability, obtained earlier by rigorous methods as the result of fairly cumbersome arguments and derivations, can be found quite simply by the use of an integral criterion.

The object of this paper is to prove the criterion, and also to determine the class of nonlinear systems and their motions for which there exist similar criteria of stability. A simplified method is given for obtaining relations which determine the phases of rotation of vibrators in synchronized motion.

1. Simplified method of finding synchronized motions and an integral criterion of stability in problems on self-synchronization of vibrators. In the solution of concrete problems on the self-synchronization of mechanical vibrators [1,2] the most difficult step is the separation of the stable motions from all the found synchronized motions.

The synchronized motions which are far from resonance can correspond to each of the real solutions of some system of transcendental equations (Equation (2.18) of [1] and Equation (2.6) of [2]) in the quantities $a_1, \ldots, a_k$ (k is the number of vibrators), which are called generating phases. The generating phases are determined by the mentioned equations to within an unessential additive constant a_0. In other words, one of the phases, for example a_k, can be assumed to be zero without loss of generality.

The process of the investigation of stability consists of the construction of an algebraic equation of degree $k - 1$, written in the form of a determinant of order $k - 1$, and of the study of the signs of the real parts of the roots of this equation, which correspond to each solution of the mentioned transcendental equations. The coefficients of the indicated determinant are computed separately for each of the various classes of problems on self-synchronization; in [1] this was done for the case of a vibrating object with one degree of freedom, while in [2] it was carried out for the case of a vibrating object in the form of a solid body performing planar parallel motion, i.e. having three degrees of freedom.

One of the interesting practical particular cases of the problem on self-synchronization is the case of identical (or almost identical) vibrators having the same (or almost the same) positive partial angular velocities.*

In all concrete problems studied by the authors of this note and related to this case, the result of the previously described process of investigation of the nature of the possible synchronized motion and of their stability has indicated that the following assertion is true.

In all synchronized motions, determined by the solutions $\alpha_1^*, \ldots, \alpha_k^*$ of the equations for the generating phases, only those will be stable which correspond to the minimum value of the mean, over the period $2\pi/\omega$, of Lagrange's function

$$\Lambda_0 = \Lambda_0(\alpha_1, \ldots, \alpha_k) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} L_0\, dt, \qquad L_0 = T_0 - \Pi_0 \tag{1.1}$$

evaluated for the auxiliary body (T_0 and Π_0 are the kinetic and potential energy, respectively, of the auxiliary body; ω is the angular velocity of the synchronized rotation of the vibrator).**

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* By identical vibrators we mean vibrators all of whose parameters, with the possible exception of the rotating moments L_r, are the same. With regard to L_r, we assume only equality of absolute values. In the case when one considers rotation of identical vibrators with the same and positive partial velocities $\omega_r = \sigma_r L_r\ (\sigma_r\omega_r)/k_r$, one has: $\omega_0 = \omega_r > 0$ and $\sigma_r L_r(\sigma_r\omega_r) - k_r\omega_0 = 0$, where $k_r = k$ is the coefficient of the rotational resistance of the rotor of the vibrator [2].

** By the auxiliary body we mean a rigid body which at every instant of time coincides with the given body on which the vibrators are installed (vibrating organ). The auxiliary body is obtained from the given solid body by adding to it the mass of all unbalanced rotors concentrated on the axes of rotation of the vibrators.

Hereby, the values of the generating phases $\alpha_1^*, \ldots, \alpha_k^*$, which correspond to the synchronized motions, can be determined up to within an additive constant from the conditions that the function $\Lambda_0(\alpha_1, \ldots, \alpha_n)$ by stable. These conditions are thus equivalent to the transcendental equations mentioned earlier.

It is important that for the application of the formulated integral criterion of stability to the considered problems on self-synchronization it is not necessary to find the exact values of the coordinates and velocities of the system of synchronized motions, but it is sufficient to restrict oneself only to the initial approximation which corresponds to the generating solution. This solution corresponds to the uniform rotation of the vibrators according to the law

$$\varphi_s = \sigma_s(\omega t + \alpha_s) \qquad (s = 1, \ldots, k) \tag{1.2}$$

(σ_s is equal to $+1$ or -1, depending on the direction of the rotation of the sth vibrator) and to the stabilized forced oscillations of the vibrating organ under the action of the disturbing forces produced by the vibrators moving in accordance with the law (1.2).

The investigation is considerably simplified also because of the fact that Expression (1.1) does not contain the kinetic and potential energies of the entire system, but does contain much simpler expressions T_0 and Π_0 which are quadratic forms of the generalized velocities and generalized coordinates determining the position of the vibrating element.

The authors do not know of any general principles or theorems of mechanics which might imply directly the above-formulated assertion. The integral criterion of stability of synchronized motions has been used for the purpose of obtaining predictions on the nature of stability of motion in complicated cases of the problem for which the solution has as yet not been obtained by the methods of Poincare and Liapunov. These predictions have all been verified by experiment.

Of great interest is the problem on the determination of the class of systems and motions for which similar integral criteria of stability might exist. We shall not go into the consideration of this general problem, but shall restrict ourselves to the illustration of the application of a simplified procedure for finding the equations for the generating phases and of the integral criterion of stability with the aid of a number of simple examples.

2. Problem of self-synchronization of two identical vibrators installed on a vibrating element with one degree of freedom. We shall investigate the synchronized motions of a given system (Fig. 1) by the method presented in Section 1.

The equation of small oscillations of the vibrating element, when the vibrators are rotating in accordance with the law

$$\varphi_1 \quad \sigma_1(\omega t+\alpha_1), \qquad \varphi_2=\sigma_2(\omega t+\alpha_2), \qquad |\sigma_1|=|\sigma_2|=1 \tag{2.1}$$

has the form

$$\ddot{x}+\lambda^2 x=\frac{F}{M}\left[\cos(\omega t+\alpha_1)+\cos(\omega t+\alpha_2)\right] \tag{2.2}$$

It is assumed here that the motion is independent of the values of the parameters σ_1 and σ_2 which determine the direction of the rotation of the vibrators.

Here, M is the mass of the system, $\lambda = \sqrt{c/M}$ is the frequency of the characteristic oscillations, c is the rigidity of the elastic elements and F is the amplitude of the disturbing forces developed by each vibrator.

The periodic solution of Equation (2.2) which corresponds to the forced stabilized oscillations is given by

$$x=\frac{F}{M(\lambda^2-\omega^2)}\left[\cos(\omega t+\alpha_1)+\cos(\omega t+\alpha_2)\right] \tag{2.3}$$

The mean, over one period, of Lagrange's function is

$$\Lambda_0=\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\left(\frac{1}{2}M\dot{x}^2-\frac{1}{2}cx^2\right)dt=\frac{1}{2}\frac{F^2}{M}\frac{\cos\alpha}{(\omega^2-\lambda^2)}+C_0 \tag{2.4}$$

where $a = a_1 - a_2$ is the relative phase shift of rotation of the vibrators, and C_0 is a quantity which depends on a.

Equating to zero the derivative $d\Lambda_0/da$, we obtain the equation $\sin a = 0$, which has two essentially different roots $(a)_1 = 0$, $(a)_2 = \Pi$.

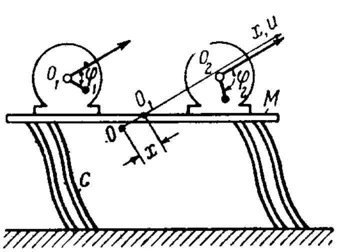

Fig. 1.

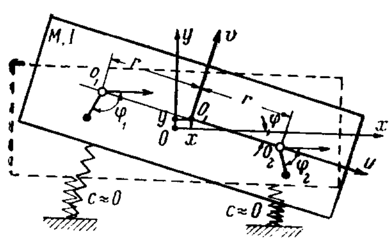

Fig. 2.

In the pre-resonance region $(\omega < \lambda)$ the first solution corresponds to a minimum, while the second solution corresponds to a maximum of the function $\Lambda_0 a$. In the post-resonance region $(\omega > \lambda)$ the value $a = (a)_1 = 0$

corresponds to a maximum, while the value $a = (a)_2 = \Pi$ corresponds to a minimum. Hence, according to the integral criterion, we can draw the following conclusions: in the pre-resonance region we have only in-phase stable synchronized motion, while in the post-resonance region we have only opposite-phase synchronized motion of the vibrators irrespective of the direction of rotation.

These results, which were obtained by the use of the integral criterion of stability, coincide completely with the results found for the given case by the methods of Poincaré and Liapunov [1].

3. Problem on the self-synchronization of two identical vibrators placed symmetrically on a softly damped vibrating element with three degrees of freedom. Suppose that the vibrating element is an absolutely rigid body which can perform planar parallel motion and which is connected to a fixed base by means of a system of damped springs. The springs will be considered to be so soft that the largest frequency of the characteristic oscillations of the body on the dampers is negligibly small as compared to the frequency of forced oscillations (Fig. 2). On the vibrating element there are placed, symmetrically to its center of gravity, two identical unbalanced vibrators with parallel axes of rotation.

The equations of motion of the vibrating element during the rotation of the vibrators according to the law (2.1) have the following form:

$$\begin{aligned} M\ddot{x} &= F\,[\cos(\omega t + \alpha_1) + \cos(\omega t + \alpha_2)] \\ M\ddot{y} &= -F\,[\sigma_1 \sin(\omega t + \alpha_1) + \sigma_2 \sin(\omega t + \alpha_2)] \\ I\ddot{\varphi} &= Fr\,[\sigma_1 \sin(\omega t + \alpha_1) - \sigma_2 \sin(\omega t + \alpha_2)] \end{aligned} \qquad (3.1)$$

Here, x and y are the coordinates of the center of gravity of the vibrating element in the fixed system of axes xOy, ϕ is the angle of rotation of the vibrating element with respect to these axes in the clockwise sense, M is the mass of the system, I is the moment of inertia of the rigid body and of the vibrators computed under the assumption that the mass of the unbalanced rotors is concentrated on their axes of rotation, r is the distance from the axes of rotation of the vibrators to the center of gravity of the body, F is, as before, the amplitude of the disturbing forces developed in each vibrator.

Because of the assumption on the softness of the elastic supports, one can assume that the potential energy of the system is zero. Hence, having determined $\dot{x}$, $\dot{y}$ and $\dot{\phi}$ for the stabilized motion by integrating Equations (3.1), we obtain

$$\Lambda_0 = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} T_0 dt = \frac{\omega}{4\pi} \int_0^{2\pi/\omega} [M(\dot{x}^2 + \dot{y}^2) + I\dot{\varphi}^2]\, dt$$
$$= \frac{F_2}{2M\omega^2} \left(1 + \sigma_1\sigma_2 - \sigma_1\sigma_2 \frac{Mr^2}{I} \right) \cos\alpha + C_1 \qquad (3.2)$$

Here, $\alpha = \alpha_1 - \alpha_2$ is, as before, the relative phase shift of rotation of the vibrators and C_1 is independent of the angle α.

Equating to zero the derivative $d\Lambda_0/d\alpha$, we obtain the equation $\sin \alpha = 0$, which has two essentially distinct roots $(\alpha)_1 = 0$ and $(\alpha)_2 = \Pi$.

It is obvious that if the vibrators rotate in the same direction ($\sigma_1\sigma_2 = 1$) the first root corresponds to a minimum of the function $\Lambda_0(\alpha)$ under the condition that

$$Mr^2 / I > 2 \qquad (3.3)$$

The second root corresponds in this case to a maximum of the function $\Lambda_0(\alpha)$. If the condition (3.3) is not satisfied, the opposite conclusion is valid.

In case the vibrators rotate in opposite directions ($\sigma_1\sigma_2 = -1$), the value $\alpha = (\alpha)_1 = 0$ corresponds always to a maximum, while the value $\alpha = (\alpha)_2 = \Pi$ will correspond to a minimum of the function $\Lambda_0(\alpha)$.

According to the integral criterion, if $\sigma_1\sigma_2 = 1$ and if condition (3.3) holds, only the in-phase motion is stable, while if (3.3) is not satisfied then the stable motion occurs when the vibrators rotate with opposite phase; if $\sigma_1\sigma_2 = -1$, only the opposite-phase rotation is stable. It is not difficult to prove that in the given case the results obtained by means of the integral criterion coincide completely with those found by the methods of Poincaré and Liapunov [2,3].

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